J. Fluid Mech. (1986), vol. 173, pp. 289–302
Printed in Great Britain

On the existence of localized rotational disturbances which propagate without change of structure in an inviscid fluid

By H. K. MOFFATT

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, UK

(Received 1 May 1986)

A wide class of solutions of the steady Euler equations, representing localized rotational disturbances imbedded in a uniform stream U_0 is inferred by considering the process of magnetic relaxation to analogous magnetostatic equilibria. These solutions, which may be regarded as generalizations of vortex rings, are characterized by their streamline topology, distinct topologies giving rise to distinct solutions.

Particular attention is paid to the class of axisymmetric solutions described by Stokes stream function $\psi(s, z)$. It is argued that the appropriate topological 'invariant' characterizing the flow is the function $V(\psi)$ representing the volume inside toroidal surfaces $\psi = \text{const.}$ in the region of closed streamlines where $\psi > 0$. This function is described as the 'signature' of the flow, and it is shown that in a certain sense, flows with different signatures are topologically distinct. The approach yields a method by which flows of arbitrary signature $V(\psi)$ may in principle be found, and the corresponding vorticity $\omega_\varphi = sF(\psi)$ calculated.

1. Introduction

In an earlier paper (Moffatt 1985, hereinafter referred to as M85), the analogy between the Euler equations for the steady flow of an inviscid incompressible fluid,

$$\boldsymbol{u} \wedge \boldsymbol{\omega} = \nabla h, \quad \boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u} = 0, \tag{1.1}$$

and the equations of magnetostatic equilibrium in a perfectly conducting fluid,

$$\boldsymbol{j} \wedge \boldsymbol{B} = \nabla p, \quad \boldsymbol{j} = \operatorname{curl} \boldsymbol{B}, \quad \nabla \cdot \boldsymbol{B} = 0, \tag{1.2}$$

has been exploited, in a manner suggested by Arnol'd (1974), with a view to establishing the existence of solutions of both problems of arbitrarily complex topology. The essence of the approach lies in the observation that the magnetostatic problem (1.2) lends itself to treatment by a relaxation technique: if an initial field $\boldsymbol{B}(\boldsymbol{x}, 0)$ of arbitrary topology does *not* satisfy the magnetostatic conditions (1.2), i.e. if

$$\operatorname{curl} \{\boldsymbol{B}(\boldsymbol{x}, 0) \wedge \operatorname{curl} \boldsymbol{B}(\boldsymbol{x}, 0)\} \not\equiv 0, \tag{1.3}$$

then the fluid will move in response to the rotational body force. If the fluid carrying the magnetic field is assumed viscous (but still perfectly conducting) then the energy of the system (initially entirely magnetic) will be dissipated by viscosity. During this process the lines of force (or '$\boldsymbol{B}$-lines') are frozen in the fluid, so that all knots and linkages of $\boldsymbol{B}$-lines are conserved. These knots and links in general impede the decay of the field, which is forced to seek an equilibrium, satisfying (1.2), and yet being 'topologically accessible' from the initial field $\boldsymbol{B}(\boldsymbol{x}, 0)$. To each such equilibrium there is then a strictly analogous solution of the Euler problem (1.1).

A characteristic feature of these solutions is that they may include tangential discontinuities of $\boldsymbol{B}$ (i.e. current sheets) or analogously of $\boldsymbol{u}$ (i.e. vortex sheets) distributed in some way within the fluid domain. These may form during the magnetic-relaxation process due to the approach of two 'magnetic surfaces', the fluid between them being squeezed out in a manner that does not involve stretching of $\boldsymbol{B}$-lines and is therefore compatible with the decrease of magnetic energy characteristic of the relaxation process. This squeezing process is generally three-dimensional in character: we shall find that under particular symmetry conditions (see §4), the process cannot occur, and so tangential discontinuities cannot form. This leads to some important conclusions concerning the existence of smooth axisymmetric Euler flows.

In M85, we supposed that the fluid was contained in a bounded domain $\mathscr{D}$, with boundary condition $\boldsymbol{B}\cdot\boldsymbol{n} = 0$ (or $\boldsymbol{u}\cdot\boldsymbol{n} = 0$) on $\partial\mathscr{D}$. In the present paper, we shall suppose that the fluid fills all space, and that the initial magnetic field in the magnetic relaxation problem has the form

$$\boldsymbol{B}(\boldsymbol{x}, 0) = \boldsymbol{B}_0 + \boldsymbol{b}_0(\boldsymbol{x}), \tag{1.4}$$

where $\boldsymbol{B}_0$ is uniform (the 'field at infinity') and $\boldsymbol{b}_0(\boldsymbol{x})$ is a smooth solenoidal field, localized in the sense that

$$r^3|\boldsymbol{b}_0(\boldsymbol{x})| < \infty \quad \text{for all } \boldsymbol{x}, \tag{1.5}$$

where $r = |\boldsymbol{x}|$. An arbitrary 'blob' of current $\boldsymbol{j}_0(\boldsymbol{x})$ confined to a finite region would, for example, produce a field $\boldsymbol{b}_0(\boldsymbol{x})$ satisfying the condition (1.5). We shall consider the relaxation of the field (1.4) towards a magnetostatic equilibrium

$$\boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x}) = \boldsymbol{B}_0 + \boldsymbol{b}^{\mathrm{E}}(\boldsymbol{x}), \tag{1.6}$$

where $\boldsymbol{b}^{\mathrm{E}}(\boldsymbol{x})$, as will become apparent, satisfies the same condition (1.5). The analogous Euler flows then have the form

$$\boldsymbol{U}^{\mathrm{E}}(\boldsymbol{x}) = \boldsymbol{U}_0 + \boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x}) \tag{1.7}$$

and this structure includes all vortex rings of known type (see, for example, Fraenkel 1970, 1972; Norbury 1973). The procedure, when restricted to axisymmetric relaxation, indicates the existence of a very wide class of vortex rings, which can be characterized by a function $V(\psi)$ representing the volume of fluid contained inside a toroidal stream surface $\psi = \text{const}$. We shall describe the function $V(\psi)$ as the 'signature' of the vortex. The signature $V(\psi)$ has topological significance, in that two axisymmetric flows of 'vortex-ring structure' characterized by Stokes stream functions ψ_1 and ψ_2 are topologically equivalent only if they have the same signature; and we shall show in effect that for every monotonic decreasing signature $V(\psi)$ $(0 < \psi < \psi_N)$ with $V(0) = V_0$, $V(\psi_N) = 0$, there exists a corresponding steady vortex-ring structure. An analogous result may be obtained for two-dimensional flows.

2. Some general features of the magnetic-relaxation problem

If the 'perturbation field' $\boldsymbol{b}_0(\boldsymbol{x})$ in (1.4) is weak relative to $\boldsymbol{B}_0$, then the field lines of $\boldsymbol{B}(\boldsymbol{x}, 0)$ will be weakly perturbed from those of $\boldsymbol{B}_0$, as indicated in figures 1 (*a*, *b*). If $\boldsymbol{b}_0(\boldsymbol{x})$ is produced by a localized displacement field $\boldsymbol{\eta}(\boldsymbol{x})$ acting on $\boldsymbol{B}_0$, i.e. if

$$\boldsymbol{b}_0(\boldsymbol{x}) = (\boldsymbol{B}_0\cdot\nabla)\,\boldsymbol{\eta} = B_0\frac{\partial}{\partial z}\boldsymbol{\eta} \tag{2.1}$$

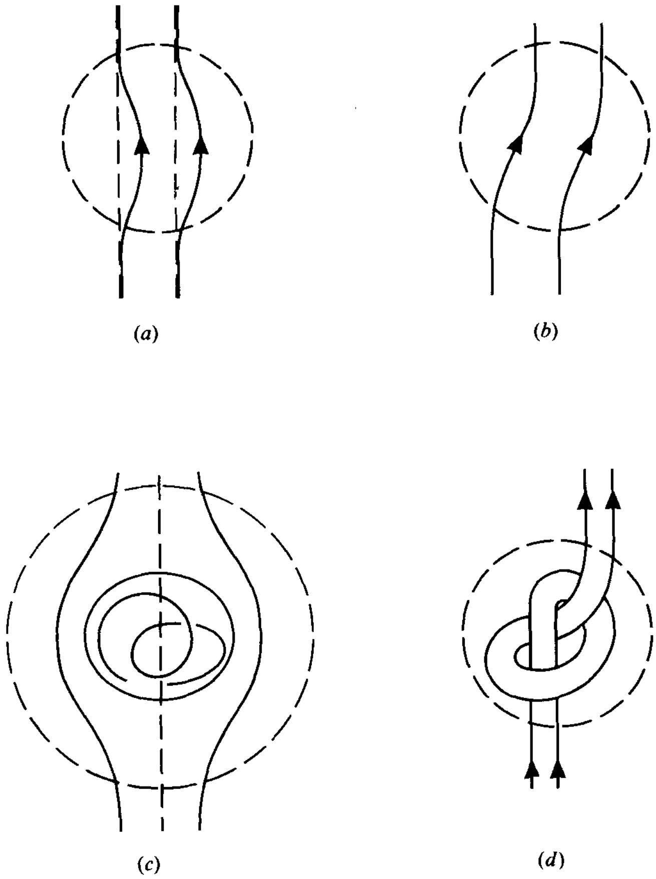


FIGURE 1. Possible topologies for the initial field $\boldsymbol{B}_0+\boldsymbol{b}_0(\boldsymbol{x})$. The disturbance is localized within the dashed circle. (*a*) Fields obtainable from localized displacement; (*b*) fields not obtainable from localized displacement; (*c*) field with region of trapped field lines; (*d*) field of complex (knotted) topology.

there is no net 'shift' of a line of force as it passes through the disturbance region (figure 1*a*). However, a general localized field $\boldsymbol{b}_0(\boldsymbol{x})$ *will* involve a net shift $\boldsymbol{\delta}(x, y)$ given, to lowest order, by

$$\boldsymbol{\delta}(x, y) = B_0^{-1} \int_{-\infty}^{\infty} (b_{0x}(\boldsymbol{x}), b_{0y}(\boldsymbol{x}), 0)\,\mathrm{d}z, \tag{2.2}$$

as indicated in figure 1 (*b*).

If $|\boldsymbol{b}_0(\boldsymbol{x})|$ is locally of the same order as $|\boldsymbol{B}_0|$ or greater, then some field lines of $\boldsymbol{B}(\boldsymbol{x}, 0)$ may be 'trapped' in the perturbed region, as indicated in figure 1 (*c*); we shall denote by $\mathscr{D}_0$ the domain within which field lines of $\boldsymbol{B}(\boldsymbol{x}, 0)$ are trapped, and by $\hat{\mathscr{D}}_0$ the exterior domain in which field lines go to $z = \pm\infty$ if followed far enough. Obviously, $\boldsymbol{B}(\boldsymbol{x}, 0)\cdot\boldsymbol{n} = 0$ on $\partial\mathscr{D}_0$, the boundary of $\mathscr{D}_0$.

There are also unpleasant possibilities like that indicated in figure 1 (*d*), in which some flux tubes are knotted as they pass through the disturbance region.

In all these cases the Lorentz force $(\nabla \wedge \boldsymbol{b}_0) \wedge \boldsymbol{B}(\boldsymbol{x}, 0)$ is in general rotational, and so motion of the fluid must ensue. Let $\boldsymbol{v}(\boldsymbol{x}, t)$, $\boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{B}_0 + \boldsymbol{b}(\boldsymbol{x}, t)$, denote the velocity and magnetic fields for $t \geqslant 0$, with initial conditions

$$\boldsymbol{v}(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{b}(\boldsymbol{x}, 0) = \boldsymbol{b}_0(\boldsymbol{x}). \tag{2.3}$$

Assuming unit fluid density, these fields evolve according to the equations

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{\nabla} \wedge (\boldsymbol{v} \wedge \boldsymbol{B}), \tag{2.4}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}p + (\boldsymbol{\nabla} \wedge \boldsymbol{b}) \wedge \boldsymbol{B} + \nu\nabla^2\boldsymbol{v}, \tag{2.5}$$

and $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, $\boldsymbol{\nabla}\cdot\boldsymbol{b} = 0$. The Lorentz force, which may be written as the divergence of the Maxwell stress tensor, integrates to zero over the whole fluid, and so the total momentum generated is zero. Thus we may safely assume that there is no Stokeslet ($O(r^{-1})$) term in the velocity field, and that

$$|\boldsymbol{v}| = O(r^{-2}), \quad |p-p_\infty| = O(r^{-3}) \quad \text{at most as } r\to\infty. \tag{2.6}$$

It is then clear from (2.4) that

$$|\boldsymbol{b}| = O(r^{-3}) \quad \text{as } r\to\infty \tag{2.7}$$

for all (finite) $t > 0$. The dominant terms of (2.5) are then all $O(r^{-4})$, and the assumptions (2.6), (2.7) are self-consistent for all $t > 0$.

We can now easily construct an energy equation. Multiplying (2.4) scalarly by $\boldsymbol{b}$, (2.5) by $\boldsymbol{v}$, adding and integrating over a large sphere V_R of radius R, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_R} \tfrac{1}{2}(\boldsymbol{b}^2+\boldsymbol{v}^2)\,\mathrm{d}V = \int_{\delta R} \boldsymbol{F}_E\cdot\boldsymbol{n}\,\mathrm{d}S - \nu\int_{V_R}(\boldsymbol{\nabla}\wedge\boldsymbol{v})^2\,\mathrm{d}V, \tag{2.8}$$

where

$$\boldsymbol{F}_E = -\boldsymbol{b}\wedge(\boldsymbol{v}\wedge\boldsymbol{B}) - \boldsymbol{v}(p-p_\infty) + \nu\boldsymbol{v}\wedge(\boldsymbol{\nabla}\wedge\boldsymbol{v}) - \tfrac{1}{2}\boldsymbol{v}v^2. \tag{2.9}$$

From (2.6) and (2.7), $|\boldsymbol{F}_E| = O(r^{-5})$ as $r\to\infty$, and so the surface contribution in (2.8) vanishes in the limit $R\to\infty$. Hence, defining

$$M(t) = \tfrac{1}{2}\int \boldsymbol{b}^2\,\mathrm{d}V, \tag{2.10}$$

$$K(t) = \tfrac{1}{2}\int \boldsymbol{v}^2\,\mathrm{d}V, \tag{2.11}$$

and

$$\Phi(t) = \nu\int(\boldsymbol{\nabla}\wedge\boldsymbol{v})^2\,\mathrm{d}V, \tag{2.12}$$

the integrals now being throughout all space, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[M(t)+K(t)] = -\Phi(t). \tag{2.13}$$

This is just as in M85, except that now $M(t)$ is the magnetic energy of the disturbance field, rather than the total field. Equation (2.13) of course implies that $M(t)+K(t)$ decreases monotonically for so long as $\Phi(t) \neq 0$.

Consider first the situations of figures 1 (a, b) in which the topology of the initial field is essentially the same as that of a uniform field. The initial disturbance then simply splits into two Alfvén waves which propagate in the directions $\pm\boldsymbol{B}_0$ and are damped by viscosity. If $B_0 L/\nu \gg 1$, where L is the scale of the initial disturbance, then we may distinguish three stages in this process (see figures 2a, b, c).

(a) $t \lesssim L/B_0$. During this initial stage, the two Alfvén waves are still overlapping, and nonlinear interaction between them may influence their structure.

(b) $L/B_0 \ll t \ll L^2\nu^{-1}$. During this intermediate stage, the Alfvén waves are

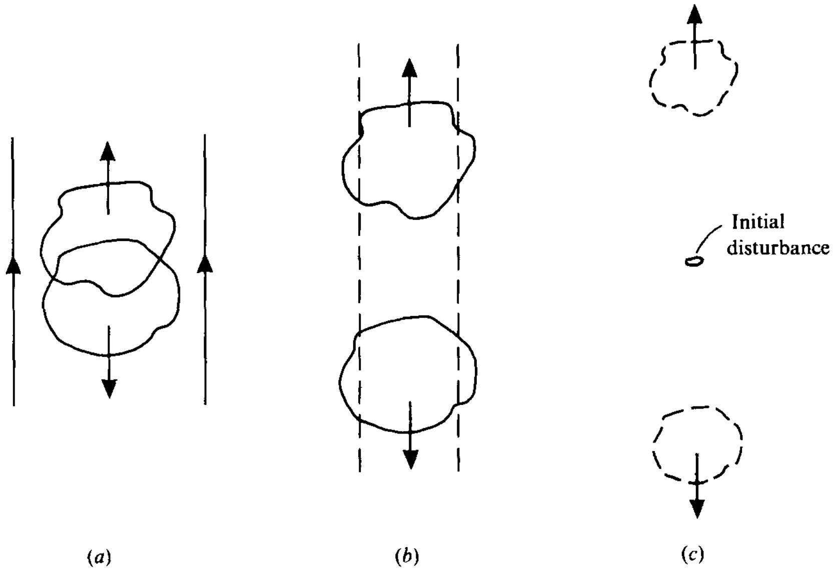


FIGURE 2. Three stages of Alfvén wave propagation: (*a*) initial stage of overlapping waves; (*b*) intermediate stage when solution is given by (2.14); (*c*) final stage when waves decay due to viscosity.

non-overlapping, and viscosity is still negligible; the exact solution of the nonlinear equations (2.4), (2.5) then takes the form

$$\left.\begin{aligned} \boldsymbol{v}(\boldsymbol{x},t) &= \tfrac{1}{2}\{\boldsymbol{f}(\boldsymbol{x}-\boldsymbol{B}_0 t)+\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{B}_0 t)\},\\ \boldsymbol{b}(\boldsymbol{x},t) &= \tfrac{1}{2}\{-\boldsymbol{f}(\boldsymbol{x}-\boldsymbol{B}_0 t)+\boldsymbol{g}(\boldsymbol{x}+\boldsymbol{B}_0 t)\},\end{aligned}\right\} \tag{2.14}$$

where the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ are determined in principle by the initial perturbation $\boldsymbol{b}_0(\boldsymbol{x})$; in the limit $|\boldsymbol{b}_0(\boldsymbol{x})|/B_0 \ll 1$, this relationship is trivial:

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{b}_0(\boldsymbol{x}), \tag{2.15}$$

but nonlinear interactions during stage (*a*) will cause departures from this result of linearized analysis.

(*c*) $t \gtrsim L^2/\nu$. During this final stage, viscosity causes diffusive spreading of the separate Alfvén waves. If this process is treated using linearized equations, it is found that the disturbance spreads relative to the 'centres of disturbance' $\boldsymbol{x} = \pm\boldsymbol{B}_0 t$ like $(\nu t)^{\frac{1}{2}}$, and $\boldsymbol{b}$ and $\boldsymbol{v}$ decay like $(\nu t)^{-2}$ (Saffman 1961). The total energy of the disturbances decays like $(\nu t)^{-\frac{5}{2}}$, and the rate of dissipation is proportional to $(\nu t)^{-\frac{7}{2}}$.

Consider now the situation of figure 1 (*c*) in which there is a finite domain $\mathscr{D}_0$ of volume V_0 in which the field lines of $\boldsymbol{B}_0+\boldsymbol{b}_0(\boldsymbol{x})$ are trapped. In the external region $\hat{\mathscr{D}}_0$, Alfvén waves may propagate to $\pm\infty$ along the $\boldsymbol{B}_0$-lines, but since the field topology is conserved, a domain $\mathscr{D}(t)$ of trapped field lines, of constant volume V_0, must survive for all $t > 0$. The shape of $\mathscr{D}(t)$ will gradually adjust itself, and as it does so Alfvén waves will continue to be generated and to propagate away in the external region $\hat{\mathscr{D}}(t)$. The total disturbance energy will decrease according to (2.13) until it attains a minimum compatible with the initial conditions.

In general, it is linkage of $\boldsymbol{B}$-lines in $\mathscr{D}(t)$ that guarantees that this minimum is non-zero. For if the topology of $\boldsymbol{B}$ in $\mathscr{D}(t)$ is trivial in the sense that each $\boldsymbol{B}$-line is an unknotted closed curve which may be shrunk to a point in $\mathscr{D}(t)$ without cutting

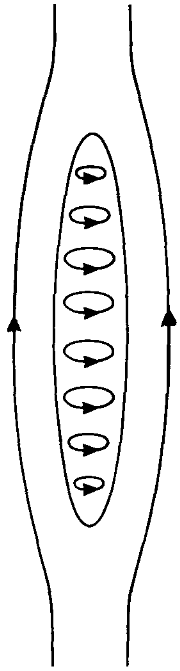


FIGURE 3. Field configuration for large t if the topology in $\mathscr{D}(t)$ is trivial.

any other $\boldsymbol{B}$-lines, then $\mathscr{D}(t)$ could relax to a long cigar-shaped region (figure 3), the field perturbation tending to zero as the length of the 'cigar' increases without limit. A measure of the linkage in $\mathscr{D}(t)$ is provided by the invariant magnetic helicity

$$\mathscr{H} = \int_{\mathscr{D}(t)} \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V, \tag{2.16}$$

where $\boldsymbol{B} = \operatorname{curl}\boldsymbol{A}$, the gauge of $\boldsymbol{A}$ being arbitrary. This suggests that we should seek a lower bound for $M(t)$ in terms of $\mathscr{H}$, as in M85 (and Arnol'd 1974), the difference now being that the volume and topology of $\mathscr{D}(t)$ are prescribed, but its shape is determined only as part of this process of energy minimization.

It will be sufficient here to give an order-of-magnitude argument for determining the minimum magnetic energy M^{E}. Suppose that the field linkage in $\mathscr{D}(t)$ is the simplest possible, namely that we have two linked flux tubes each of volume $\frac{1}{2}V_0$ and each carrying flux Φ. The helicity associated with this linkage (Moffatt 1969) is given by

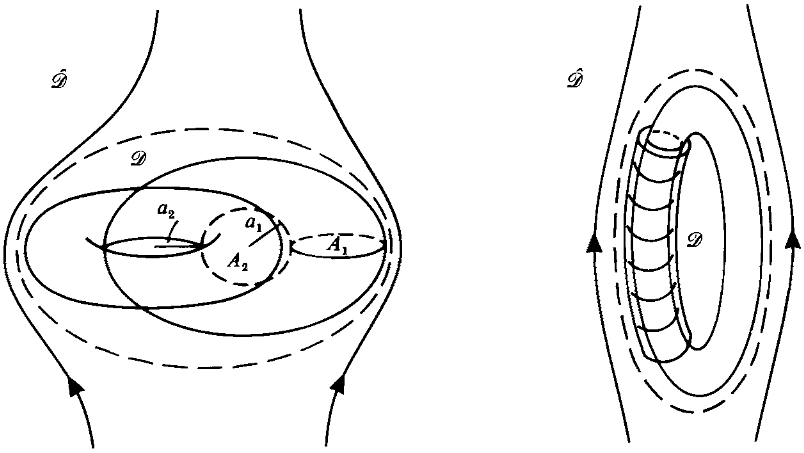
$$\mathscr{H} = \pm 2\Phi^2, \tag{2.17}$$

the sign depending on whether the linkage is right-handed or left-handed. If these flux tubes are tori (figure 4*a*) with sectional areas A_1, A_2 and inner radii $a_1 \sim V_0/A_1$, $a_2 \sim V_0/A_2$ (where constants of order unity are neglected) then the magnetic energy in $\mathscr{D}(t)$ is

$$M^{-} \sim V_0\,\Phi^2\left(\frac{1}{A_1^2}+\frac{1}{A_2^2}\right). \tag{2.18}$$

This is minimized when $a_1^2 \sim A_2$ and $a_2^2 \sim A_1$, since then the flux tubes are 'maximally contracted' as in figure 4(*a*); hence $A_1 \sim A_2 \sim V_0^{\frac{2}{3}}$ and so, from (2.18),

$$M^{-}_{\min} \sim \Phi^2 V_0^{-\frac{1}{3}} \sim |\mathscr{H}|\,V_0^{-\frac{1}{3}}. \tag{2.19}$$



(*a*) (*b*)

FIGURE 4. Linked flux tubes imbedded in a uniform field; the perturbation magnetic energy has a minimum determined by the magnetic helicity in the region $\mathscr{D}$ of trapped field lines.

However, there is also a contribution, M^+ say, to M from the external region $\hat{\mathscr{D}}$. For any given shape of $\mathscr{D}$, the perturbation field $\boldsymbol{b}^{\mathrm{E}}(\boldsymbol{x})$ of minimum energy in $\hat{\mathscr{D}}$ satisfying

$$\left.\begin{aligned} \boldsymbol{b}^{\mathrm{E}}\cdot\boldsymbol{n} &= -\boldsymbol{B}_0\cdot\boldsymbol{n} \quad \text{on } \partial\mathscr{D}, \\ \boldsymbol{b}^{\mathrm{E}} &\rightarrow 0 \quad \text{as } |\boldsymbol{x}|\rightarrow\infty \end{aligned}\right\} \tag{2.20}$$

is the unique potential field $\boldsymbol{b}^{\mathrm{E}} = \nabla\varphi^{\mathrm{E}}$, with

$$\left.\begin{aligned} \nabla^2\varphi^{\mathrm{E}} &= 0 \quad \text{in } \hat{\mathscr{D}}, \\ \frac{\partial\varphi^{\mathrm{E}}}{\partial n} &= -B_0\cdot\boldsymbol{n} \quad \text{on } \partial\hat{\mathscr{D}}. \end{aligned}\right\} \tag{2.21}$$

If the dimensions of $\mathscr{D}$ parallel and perpendicular to $\boldsymbol{B}_0$ are both $O(V_0^{\frac{1}{3}})$, then clearly $|\boldsymbol{b}^{\mathrm{E}}| = O(B_0)$ and so $M^+ \sim B_0^2 V_0$. If however, $|\mathscr{H}|$ is weak, then $\mathscr{D}$ may become cigar shaped with dimensions of order $\epsilon^{-2}V_0^{\frac{1}{3}}$ and $\epsilon V_0^{\frac{1}{3}}$ ($\epsilon \ll 1$) parallel and perpendicular to $\boldsymbol{B}_0$ respectively (figure 4*b*). In this case, $|\boldsymbol{b}^{\mathrm{E}}| \sim \epsilon\boldsymbol{B}_0$ in $\hat{\mathscr{D}}$ so that

$$M^+ \sim \epsilon^2 B_0^2 V_0. \tag{2.22}$$

Moreover we then have $A_1 \sim \epsilon^2 V_0^{\frac{2}{3}}$, $A_2 \sim \epsilon^{-2} V_0^{\frac{2}{3}}$ so that

$$M^- \sim V_0^{-\frac{1}{3}}|\mathscr{H}|\,\epsilon^{-4}. \tag{2.23}$$

Hence $M = M^+ + M^-$ is minimized when

$$\epsilon^6 \sim |\mathscr{H}|/B_0^2\, V_0^{\frac{4}{3}} \tag{2.24}$$

and then

$$M_{\min} \sim |\mathscr{H}|^{\frac{1}{3}} B_0^{\frac{4}{3}}\, V_0^{\frac{5}{9}}. \tag{2.25}$$

This is fact gives the correct order of magnitude of $M_{\min}$ whether $\epsilon = O(1)$ or $\epsilon \ll 1$.

Thus, if $|\mathscr{H}| \neq 0$, there is a lower bound (> 0) for the magnetic energy and so, *a fortiori*, for $M(t)+K(t)$. Since $M(t)+K(t)$ is monotonic decreasing and bounded below in this way, this total energy must tend to a constant as $t\rightarrow\infty$, and so from (2.13), $\Phi(t)\rightarrow 0$ as $t\rightarrow\infty$; hence also, $K(t)\rightarrow 0$, $M(t)\rightarrow M^{\mathrm{E}}(>0)$ as $t\rightarrow\infty$. As noted already, the rate of dissipation associated with the emitted Alfvén waves decays like

$t^{-\frac{2}{3}}$; but now we see that the total rate of dissipation including that in the neighbourhood of $\mathscr{D}(t)$ must tend to zero also as $t \to \infty$.

As in M85, it now follows that, since viscosity will surely prevent the appearance of singularities in $\boldsymbol{v}$, we must have $\boldsymbol{v} \to 0$ as $t \to \infty$, and $\boldsymbol{B} \to \boldsymbol{B}^{\mathrm{E}}(\boldsymbol{x}) = \boldsymbol{B}_0 + \boldsymbol{b}^{\mathrm{E}}(\boldsymbol{x})$ where, from (2.5),

$$\boldsymbol{j}^{\mathrm{E}} \wedge \boldsymbol{B}^{\mathrm{E}} = \boldsymbol{\nabla} p^{\mathrm{E}}, \quad \boldsymbol{j}^{\mathrm{E}} = \operatorname{curl} \boldsymbol{B}^{\mathrm{E}}, \tag{2.26}$$

i.e. $\boldsymbol{B}^{\mathrm{E}}$ is a magnetostatic field, which by its construction is topologically accessible from the initial field $\boldsymbol{B}(\boldsymbol{x}, 0)$, and which has a domain $\mathscr{D}^{\mathrm{E}}$ (of volume V_0) of trapped field lines.

We have already noted that the field of minimum energy is a potential field in the external region $\hat{\mathscr{D}}^{\mathrm{E}}$. This may be seen directly from (2.26): for

$$\boldsymbol{B}^{\mathrm{E}} \cdot \boldsymbol{\nabla} p^{\mathrm{E}} = 0, \tag{2.27}$$

so that p^{E} is constant on $\boldsymbol{B}^{\mathrm{E}}$-lines and therefore constant throughout $\hat{\mathscr{D}}^{\mathrm{E}}$, being uniform at infinity. Hence

$$\boldsymbol{j}^{\mathrm{E}} = \alpha(\boldsymbol{x}) \boldsymbol{B}^{\mathrm{E}} \quad \text{in } \hat{\mathscr{D}}^{\mathrm{E}} \tag{2.28}$$

with

$$\boldsymbol{B}^{\mathrm{E}} \cdot \boldsymbol{\nabla} \alpha = 0. \tag{2.29}$$

Hence α is also constant on $\boldsymbol{B}^{\mathrm{E}}$-lines and therefore zero throughout $\hat{\mathscr{D}}^{\mathrm{E}}$, being zero at infinity. Hence $\boldsymbol{j}^{\mathrm{E}} \equiv 0$ in $\hat{\mathscr{D}}^{\mathrm{E}}$ as expected.

3. The analogous Euler flows

The analogy between the problems (1.1) and (1.2) now allows us to deduce the following result. Let $\boldsymbol{u}_0(\boldsymbol{x})$ be an arbitrary localized perturbation superposed on a uniform stream $\boldsymbol{U}_0$, such that there is a domain $\mathscr{D}_0$ of finite volume V_0 in which the streamlines of $\boldsymbol{U}_0 + \boldsymbol{u}_0(\boldsymbol{x})$ are trapped and such that the streamlines are unknotted in $\hat{\mathscr{D}}_0$. Then there exists an Euler flow $\boldsymbol{U}(\boldsymbol{x}) = \boldsymbol{U}_0 + \boldsymbol{u}^{\mathrm{E}}(\boldsymbol{x})$ with the following properties: (i) there exists a domain $\mathscr{D}^{\mathrm{E}}$ with volume V_0 and the same topology as $\mathscr{D}_0$ such that the streamlines of $\boldsymbol{U}(\boldsymbol{x})$ are trapped in $\mathscr{D}^{\mathrm{E}}$; (ii) the streamlines of $\boldsymbol{U}(\boldsymbol{x})$ in $\mathscr{D}^{\mathrm{E}}$ are topologically accessible from the streamlines of $\boldsymbol{U}_0 + \boldsymbol{u}_0(\boldsymbol{x})$ in $\mathscr{D}_0$; (iii) the flow $\boldsymbol{U}(\boldsymbol{x})$ is irrotational in the external domain $\hat{\mathscr{D}}^{\mathrm{E}}$, i.e. the vorticity

$$\boldsymbol{\omega}^{\mathrm{E}} = \operatorname{curl} \boldsymbol{U} = \operatorname{curl} \boldsymbol{u}^{\mathrm{E}} \tag{3.1}$$

is confined to $\mathscr{D}^{\mathrm{E}}$.

In a frame of reference moving with velocity $\boldsymbol{U}_0$, i.e. with the fluid at infinity, the blob of vorticity $\boldsymbol{\omega}^{\mathrm{E}}$ propagates without change of structure with velocity $-\boldsymbol{U}_0$. It would appear that there is a very wide class of solutions of the Euler equations of this kind. Clearly we are dealing with a generalization of the vortex ring, for which the term 'vorton' may be appropriate.† We emphasize that in general these solutions will have vortex sheets imbedded within the region $\mathscr{D}^{\mathrm{E}}$, and in this case are likely to be unstable. We now turn however to situations in which symmetry conditions exclude the appearance of tangential discontinuities in the magnetic-relaxation problem and so equally of vortex sheets in the analogous Euler flows.

† The term 'vorton' has been used in a more specific context by Novikov (1984); there is however little possibility of confusion in using it also in the present more general context.

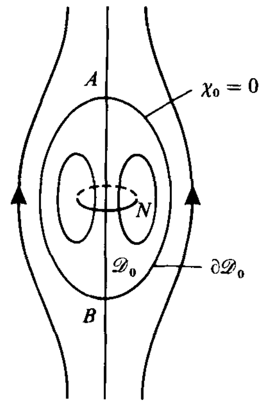


FIGURE 5. Topology of initial field $\boldsymbol{B}(\boldsymbol{x}, 0)$ for axisymmetric relaxation.

4. Axisymmetric configurations

Suppose that the initial magnetic field $\boldsymbol{B}(\boldsymbol{x}, 0) = \boldsymbol{B}_0 + \boldsymbol{b}_0(\boldsymbol{x})$ in the magnetic relaxation problem is axisymmetric about the axis Oz, and has the topology indicated in figure 5, i.e. there are only two hyperbolic neutral points ('saddles'), both on the axis of symmetry (A and B), and one elliptic neutral point N (a 'centre') in the meridian plane. In cylindrical polar coordinates (s, φ, z) this field may be expressed in terms of a flux function $\chi_0(s, z)$ analogous to Stokes' stream function:

$$\boldsymbol{B}(\boldsymbol{x}, 0) = \left(\frac{1}{s}\frac{\partial \chi_0}{\partial z}, 0, -\frac{1}{s}\frac{\partial \chi_0}{\partial s}\right). \tag{4.1}$$

The lines of force are given by $\chi_0(s, z) = \text{const.}$, and in particular the surface $\partial\mathscr{D}_0$ may be taken to be $\chi_0(s, z) = 0$. Obviously

$$\chi_0 \sim -\tfrac{1}{2}B_0 s^2 \quad \text{as } |\boldsymbol{x}| \to \infty. \tag{4.2}$$

Hence (taking $B_0 > 0$),

$$\chi_0 \begin{cases} \leqslant 0 & \text{in } \hat{\mathscr{D}}_0, \\ \geqslant 0 & \text{in } \mathscr{D}_0. \end{cases} \tag{4.3}$$

χ_0 is maximal at N, and has no other extrema in $s > 0$, $-\infty < z < \infty$.

Suppose now that we allow this field to relax, the fluid being viscous and perfectly conducting as in §2. Obviously the motion is axisymmetric (provided non-axisymmetric instabilities are ignored), and the field at time t is given by

$$\boldsymbol{B}(\boldsymbol{x}, t) = \left(\frac{1}{s}\frac{\partial \chi}{\partial z}, 0, -\frac{1}{s}\frac{\partial \chi}{\partial s}\right), \tag{4.4}$$

where $\chi(s, z, t)$ satisfies

$$\frac{\mathrm{D}\chi}{\mathrm{D}t} = 0, \quad \chi(s, z, 0) = \chi_0(s, z), \tag{4.5}$$

i.e. the flux surfaces $\chi = \text{const.}$ move with the fluid. In particular, the volume $V(\chi)$ contained inside each toroidal flux surface $\chi = \text{const.}$ $(\chi > 0)$ remains constant, by virtue of the incompressibility of the fluid. Thus the function $V(\chi)$ $(0 < \chi < \chi_N)$ is a characteristic property of the field which is invariant throughout the relaxation process; and $V(\chi)$ is a topological invariant, in the sense that a necessary condition

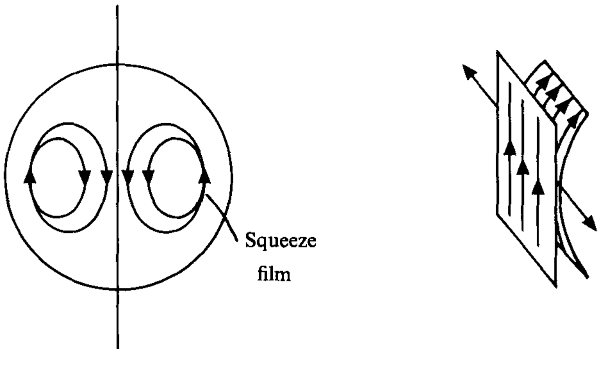


FIGURE 6. (*a*) Tori, initially separate, *cannot* come in contact as illustrated, since this would involve unbounded increase of magnetic energy; (*b*) by contrast, in three-dimensional configurations, surfaces of discontinuity can form at a squeeze film in which fluid is ejected transverse to the magnetic field.

for two fields characterized by flux maxima χ_{N1}, χ_{N2} and functions $V_1(\chi)$, $V_2(\chi)$ to be topologically accessible,† one from the other, is that

$$\left.\begin{aligned} \chi_{N1} = \chi_{N2} = \chi_N, \text{ say,} \\ V_1(\chi) = V_2(\chi) \quad (0 < \chi < \chi_N). \end{aligned}\right\} \tag{4.6}$$

Suppose now that $\chi_N > \chi_1 > \chi_2 > 0$, so that the surface $\chi = \chi_1$ is a torus nested inside the torus $\chi = \chi_2$. These two tori can never come in contact in an axisymmetric motion since, if they were to do so as indicated in figure 6(*a*), this would involve infinite stretching of $\boldsymbol{B}$-lines in the squeeze film in the final stage of approach, and this would require unbounded increase of magnetic energy which is not possible according to (2.13). Since this argument applies to any two values of χ_1, χ_2 in the range $(0, \chi_N)$ it follows that flux surfaces that are initially separate can never come together; hence there is no mechanism by which a tangential discontinuity of $\boldsymbol{B}$ may form. This is to be contrasted with the fully three-dimensional situation of §2, in which two magnetic surfaces may approach each other over a finite area, the fluid being squeezed out in a direction transverse to the field in a manner which does not involve field-line stretching (figure 6*b*). The axisymmetric constraint prevents this type of behaviour. (There appears no reason however why weaker discontinuities of the *gradient* of $\boldsymbol{B}$ should not form.)

We are thus driven to the following remarkable conclusion, which we now state in the context of the analogous Euler-flow situation (see figure 7): with cylindrical polar coordinates (s, φ, z), let $\psi_0(s, z)$ be an arbitrary C^1 function with the following properties:

(i) $\psi_0(s, z) \sim -\frac{1}{2}U_0 s^2 \quad (U_0 > 0) \quad \text{as } s^2+z^2 \to \infty$;

(ii) $\psi_0(s, z) \geqslant 0$ in a compact axisymmetric connected domain $\mathscr{D}_0$, and $\psi_0(s, z) \leqslant 0$ in the exterior domain $\hat{\mathscr{D}}_0$;

(iii) ψ_0 has a single maximum $\psi_N(>0)$ at a point $N \in \mathscr{D}_0$, each point of the circle $s = s_N$, $z = z_N$ then being an elliptic stagnation point of the corresponding flow.

† Here, we continue to use the term 'topologically accessible' in the sense defined in M85.

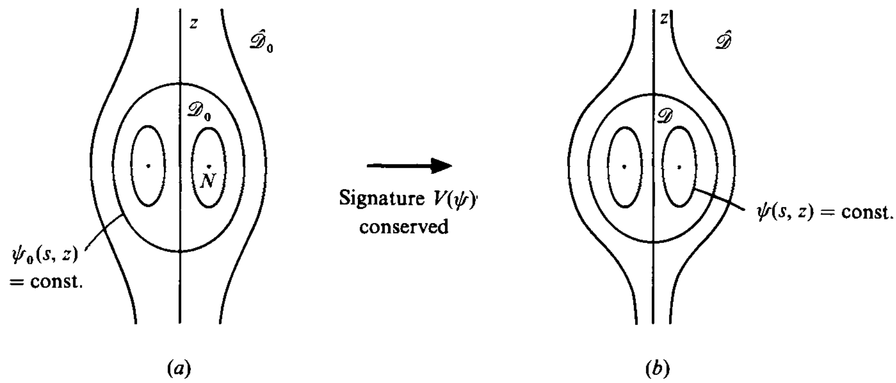


FIGURE 7. Relation between (*a*) the 'arbitrary' flow $\psi_0(s, z)$ with signature $V(\psi)$ and (*b*) the inferred Euler flow with the same signature.

For $0 \leqslant \Psi \leqslant \psi_N$, let $V(\Psi)$ be the volume inside the torus

$$\psi_0(s, z) = \Psi,$$

so that evidently $V(\Psi)$ is monotonic decreasing with

$$\left.\begin{aligned} V(\psi_N) &= 0, \\ V(0) &= V_0 \quad (\text{the volume of } \mathscr{D}_0). \end{aligned}\right\}$$

Then there exists an Euler flow represented by a C^1 Stokes stream function $\psi(s, z)$ with similar properties, namely

(i) $\psi(s, z) \sim -\frac{1}{2}U_0\, s^2 \quad \text{as } s^2+z^2 \to \infty$;

(ii) $\psi(s, z) \geqslant 0$ in a compact axisymmetric connected domain $\mathscr{D}$ having the same volume V_0 as $\mathscr{D}_0$, and $\psi(s, z) \leqslant 0$ in the exterior domain $\hat{\mathscr{D}}$;

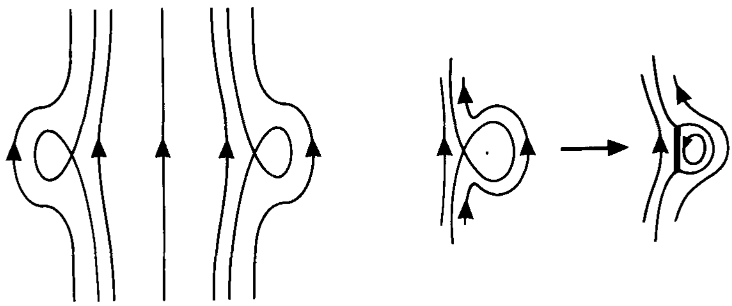
(iii) ψ has a single maximum ψ_N within $\mathscr{D}$;

(iv) the volume–flux relation $V = V(\Psi)$ is the same for the Euler flow $\psi(s, z)$ as for the 'arbitrary' flow $\psi_0(s, z)$.

The importance of the function $V(\Psi)$ in characterizing axisymmetric flows of this kind merits recognition through appropriate terminology: we shall describe $V(\Psi)$ as the 'signature' of the flow. Since obviously there is an uncountable infinity of possible signatures and since two flows are topologically accessible one from another only if they have the same signature, it follows that there is an uncountable infinity of topologically distinct Euler flows of vortex-ring character, which propagate with a given velocity U_0. For each such flow, the velocity field is continuous,† but the vorticity field may have one or more surfaces of discontinuity or even (conceivably) surfaces of discontinuity that are densely distributed in certain subdomains of $\mathscr{D}$. Surfaces of discontinuity of vorticity need cause no surprise in this context, since even the well-known spherical vortex of Hill (1894) exhibits this feature.

We have noted in the above discussion that, in the magnetic-relaxation problem, the initial field $\boldsymbol{B}_0$ is assumed to have only two hyperbolic neutral points A and B on the axis of symmetry; if however $\boldsymbol{B}_0$ has one or more hyperbolic neutral points *off* the axis of symmetry (as for example in the topology of figure 8*a*) then a tangential discontinuity of $\boldsymbol{B}$ may develop during relaxation without violating the

† Here, we need suppose merely that $V(\Psi)$ and its inverse $\Psi(V)$ are continuous.



(a) (b)

FIGURE 8. (a) Possible topology with hyperbolic neutral point off axis of symmetry; (b) possible formation of tangential discontinuity near hyperbolic neutral point during the process of relaxation; the fluid is squeezed out of the acute angle at the neutral point under the action of the local Lorentz force, and there is no topological impediment to the formation of a current-sheet discontinuity as $t \to \infty$, as illustrated.

constraint of bounded magnetic energy, as illustrated by the sketch of figure 8(b). The streamlines associated with a circular line vortex of radius a and very small core radius ($\lesssim 0.01a$) have this topology (see, for example, Batchelor 1967, p. 525).

5. Discussion

In this paper we have extended the technique suggested by Arnol'd (1974) and developed by Moffatt (1985), whereby magnetic relaxation in a perfectly conducting but viscous fluid is used to demonstrate the existence of magnetostatic equilibria, and hence of analogous Euler flows, of arbitrarily prescribed topology. In the present paper we have focused on situations in which the 'initial field' $\boldsymbol{B}(\boldsymbol{x}, 0)$ of the magnetic-relaxation problem settles down to a uniform field $\boldsymbol{B}_0$ at infinity. The relaxed equilibrium field $\boldsymbol{B}_0 + \boldsymbol{b}^{\mathrm{E}}(\boldsymbol{x})$ then has the property that the disturbance $\boldsymbol{b}^{\mathrm{E}}(\boldsymbol{x})$ is rotational only in a finite region $\mathscr{D}^{\mathrm{E}}$ within which the field lines of $\boldsymbol{B}_0 + \boldsymbol{b}^{\mathrm{E}}$ are trapped. The analogous Euler flow $\boldsymbol{U}_0 + \boldsymbol{u}^{\mathrm{E}}$ then has a similar property, and the vorticity field $\boldsymbol{w}^{\mathrm{E}} = \mathrm{curl}\, \boldsymbol{u}^{\mathrm{E}}$ then propagates without change of shape with velocity $-\boldsymbol{U}_0$ relative to the fluid 'at infinity'. This 'vorton' is a generalization of the familiar vortex ring; in general the velocity field in such a vorton may have tangential discontinuities.

When attention is limited to axisymmetric flows described by a Stokes stream function $\psi(s, z)$ we have shown that, for every 'signature' $V(\psi)$ $(0 < \psi < \psi_N)$ representing the volume inside the torus $\psi = \text{const.}$ there exists a vortex ring propagating with a given velocity $-\boldsymbol{U}_0$, and that flows of different signatures are topologically distinct in a sense that is best appreciated with reference to the analogous magnetostatic equilibria: no continuous deformation can convert one to the other.

An important property of the axisymmetric flow is that, if the reference flow $\boldsymbol{U}_0(\boldsymbol{x})$ (with stream function $\psi_0(s, z)$) is continuous, then the derived Euler flow $\boldsymbol{U}^{\mathrm{E}}(\boldsymbol{x})$ is also continuous; these are in fact related by the Cauchy equation

$$U_i^{\mathrm{E}}(\boldsymbol{X}) = U_{0j}(\boldsymbol{x}) \frac{\partial X_i}{\partial x_j},$$

where $\boldsymbol{x} \rightarrow \boldsymbol{X}(\boldsymbol{x})$ is the net particle displacement associated with the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ $(0 < t < \infty)$ in the corresponding magnetic relaxation problem. Since toroidal surfaces cannot come together in this relaxation process, under the constraint of axisymmetry, the mapping $\boldsymbol{x} \rightarrow \boldsymbol{X}(\boldsymbol{x})$ is continuous, and it is in fact a homeomorphism. The flow $\boldsymbol{U}^{\mathrm{E}}(\mathrm{x})$ is thus topologically equivalent in a strict sense to the flow $\boldsymbol{U}_0(\boldsymbol{x})$.

It is of course well known that in any steady axisymmetric flow, the azimuthal vorticity

$$\omega_{\varphi} = -(s^{-1}\psi_s)_s - s^{-1}\psi_{zz} \tag{5.1}$$

must satisfy

$$\omega_{\varphi} = sF(\psi) \tag{5.2}$$

for some function $F(\psi)$, i.e. ω_{φ}/s is constant on streamlines. For any given function $F(\psi)$ it is generally a very difficult matter to find a corresponding $\psi(s, z)$ or indeed to show that any such solution exists (see, for example, Friedman & Turkington 1981). By contrast, the technique suggested in this paper indicates that for arbitrary signature $V(\psi)$ $(0 < \psi < \psi_N)$, a vortex-ring-type solution exists, and the corresponding vorticity distribution $sF(\psi)$ is then implicitly determined. Unfortunately it is difficult to obtain an explicit relationship between the functions $V(\psi)$ and $F(\psi)$; and consideration of even the simplest case of Hill's vortex (for which $F(\psi) \propto \psi$) indicates that the relation between F and V may in general be quite complex. Numerical determination of flows characterized by signature $V(\psi)$ and of the corresponding vorticity field $sF(\psi)$ should nevertheless be straightforward.

The question perhaps remains as to how an initial flow $\psi_0(s, z)$ with a given signature $V(\psi)$ is to be constructed. A possible procedure is as follows: let $\psi_{\mathrm{H}}(s, z)$ represent Hill's vortex, with $\psi_{\mathrm{H}} = 0$ on $s^2 + z^2 = a^2$, where $\frac{4}{3}\pi a^3 = V(0)$, and denote the signature of this vortex by $V_{\mathrm{H}}(\psi)$, $(\psi > 0)$, an easily computed function. Let

$$\psi_0(s, z) = G(\psi_{\mathrm{H}}(s, z)), \tag{5.3}$$

where G is a function to be determined, satisfying $G(\psi_{\mathrm{H}}) \equiv \psi_{\mathrm{H}}$ for $\psi_{\mathrm{H}} < 0$. The streamlines of the flow ψ_0 then coincide with the streamlines of the flow ψ_{H} and so the signatures are related by

$$V_{\mathrm{H}}(\psi) = V(G(\psi)). \tag{5.4}$$

Hence the required signature $V(\psi)$ may be achieved by choosing

$$G(\psi) = V^{-1}\{V_{\mathrm{H}}(\psi)\}. \tag{5.5}$$

REFERENCES

ARNOL'D, V. 1974 The asymptotic Hopf invariant and its applications. In *Proc. Summer School in Differential Equations*, Armenian SSR Acad. Sci. (in Russian).

BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

FRAENKEL, L. E. 1970 On steady vortex rings of small cross-section in an ideal fluid. *Proc. R. Soc. Lond.* A **316**, 29–62.

FRAENKEL, L. E. 1972 Examples of steady vortex rings of small cross-section in an ideal fluid. *J. Fluid Mech.* **51**, 119–135.

FRIEDMAN, A. & TURKINGTON, B. 1981 Vortex rings: existence and asymptotic estimates. *Trans. Am. Math. Soc.* **268**, 1–37.

HILL, M. J. M. 1894 On a spherical vortex. *Phil. Trans. R. Soc. Lond.* A **185**, 213–245.

MOFFATT, H. K. 1969 The degree of knottedness of tangled vortex lines. *J. Fluid Mech.* **35**, 117–129.

MOFFATT, H. K. 1985 Magnetostatic equilibria and analogous Euler flows of arbitrarily complex topology. Part 1. Fundamentals. *J. Fluid Mech.* **159**, 359–378.

NORBURY, J. 1973 A family of steady vortex rings. *J. Fluid Mech.* **57**, 417–431.

NOVIKOV, E. A. 1984 Generalised dynamics of three dimensional vortex singularities. In *Turbulence and Chaotic Phenomena in Fluids* (ed. T. Tatsumi), pp. 259–263. North-Holland.

SAFFMAN, P. G. 1961 The final stage of decay of a localized disturbance in a conducting fluid in a uniform magnetic field. *Q. J. Mech. Appl. Maths* **14**, 20.